Time Operator for the Quantum Harmonic Oscillator: Resolution of an Apparent Paradox

Alex Granik* and H.Ralph Lewis†

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Abstract

An apparent paradox is resolved that concerns the existence of time operators which have been derived for the quantum harmonic oscillator. There is an apparent paradox because, although a time operator is canonically conjugate to the Hamiltonian, it has been asserted that no operator exists that is canonically conjugate to the Hamiltonian. In order to resolve the apparent paradox, we work in a representation where the phase operator is diagonal. The boundary condition on wave functions is such that they be periodic in the phase variable, which is related to the (continuous) eigenvalue of the time operator. Matrix elements of the commutator of the time operator with the Hamiltonian involve the phase variable itself in addition to periodic functions of the phase variable. The Hamiltonian is not hermitian when operating in space that includes the phase variable itself. The apparent paradox is resolved when this non-hermeticity is taken into account correctly in the evaluation of matrix elements of the commutation relation.

*Physics Department,University of the Pacific,Stockton,CA. 95211
†Dartmouth College,Dept.of Physics and Astronomy,NH 03755

1 Introduction

In the space spanned by the eigenstates of the coordinates operator q and the momentum operator p, we consider the quantum harmonic oscillator described by the following Hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2 \tag{1}$$

and we consider a time operator χ, which is an operator conjugate to the Hamiltonian in that space:

$$[\chi, H] = i\hbar \tag{2}$$

where $\chi = \chi(q,p)$ is a function of the operators(q,p), but is not an explicit function of the time t. For an eigenstate of H, the(continuous) eigenvalue of χ is an angle variable whose period is energy-dependent and whose domain is $(-\infty, \infty)$. It is convenient to introduce a phase operator ϕ by [1]

$$\omega\chi = \frac{\pi}{2}\mathit{1} - \phi + G(H) \tag{3}$$

where $G(H)$ is some function of H. For an eigenstate of H, the (continuous) eigenvalue of ϕ, which we denote by φ and call the phase variable, is also an angle whose domain is $(-\infty, \infty)$; but its period is the constant 2π.

Aharanov and Bohm [2] discussed the time operator for the case of a free particle. A time operator for the harmonic oscillator was derived by Bender and Dunne [3]-[5] and was re-derived using similar considerations by Lewis, Lawrence, and Harris [1], who were motivated by its application as a starting point of a perturbation theory for the construction of invariant quantum operators [6]

Susskind and Glogower [7] showed that a time operator for the harmonic oscillator does not exist in the Hilbert space of energy eigenstates, and they proposed a pair of alternate operators for use in this Hilbert space. Cannata and Ferrari [8, 9] also proposed an alternate operator fro use in the Hilbert space. In a Comment on the paper by Lewis *et al*, Smith and Vaccaro [10] disputed the existence of a time operator for the harmonic oscillator in any space by pointing at an apparent inconsistency. They incorrectly asserted that, because the eigenvalue spectrum of H is discrete, the commutation relation (2) does not have a representation in the Hilbert space of the eigenstates of H. Although Lewis *et al* denied the validity of this apparent inconsistency

in a Reply [11], they did not address the issue in detail. The existence of time operators that have been derived , together with the apparent inconsistency , constitutes an apparent paradox. This apparent inconsistency is addressed in this letter. It is removed by correctly taking into account the non-hermeticity of the Hamiltonian in a space that contains the phase variable itself in addition to periodic functions of the phase variable. As a result, the commutation relation (2) is seen to have a representation in the Hilbert space, and the apparent paradox is thus resolved. The present authors believe that a time operator for the harmonic oscillator can also be derived simply and directly from the classical time function by adapting an Moyal quantization [12]. This derivation is left for the future publication.

The apparent inconsistency raised by Smith and Vaccarots1 is based on the assertion that the relation

$$< m|[B,H]|n >=< m|B|n > (n-m)\hbar\omega \tag{4}$$

holds for any operator B, where

$$H|n >= (n+\frac{1}{2})\hbar\omega|n > \tag{5}$$

Relation (4) is correct if operation by B on any eigenstate of H generates a state that lies in the space spanned by the eigenstates of h. However B does not have that property if it is conjugate to H, as in the case with the time operator χ. Then, relation (4) is incorrect. This can be understood clearly by working in the phase variable representation, *i.e.,* in the representation of the eigenstates of φ; in that representation both φ and χ are diagonal. The action of φ on one of its eigenstates is simply multiplication by the eigenvalue; and according to (3), the action of χ on an eigenstate of φ is multiplication by a linear function of φ. Both φ and χ are hermitian in the full space of the eigenstates of φ and in every subspace thereof. The reason why (4) is incorrect when applied to $< m|[\chi,H]|n >$ is that H is *not hermitian* in the full space of the eigenstates of φ. The physical boundary condition on wave functions is that they be periodic functions of φ. In a space of periodic functions of φ, the Hamiltonian H is hermitian. However in a space that includes aperiodic functions of φ, like variable φ itself , H is not hermitian. We evaluate $< m|[\chi,H]|n >$ as follows

$$\begin{aligned} < m|[\chi,H]|n > &=< m|\chi H|n > - < m|H\chi|n > \\ &= (n+\tfrac{1}{2})\hbar\omega < m|\chi|n > - < m|H\chi|n >, \end{aligned} \tag{6}$$

where $< m|[\chi, H]|n >$ has been evaluated using (5). If H *were hermitian* then also the last term in (6) could be evaluated directly using (5):

$$\begin{array}{c} < m|H\chi|n >=< n|\chi^\dagger H^\dagger|n >^*= (m+\frac{1}{2})\hbar\omega < n|\chi^\dagger|m >^*= \\ (m+\frac{1}{2})\hbar\omega < m|\chi|n > \end{array} \tag{7}$$

This would give an evaluation of $< m|[\chi, H]|n >$ in agreement with (4). However, H cannot be considered hermitian in the evaluation of $< m|[\chi, H]|n >$ because the presence of χ in $< m|[\chi, H]|n >$ leads to aperiodic functions of φ. Therefore relation (4) is invalid in this case. The is consistent with the discussion in the introductory section of a paper by Cannata and Ferrari [9], where it is demonstrated that relation (4) holds for any operator B whose action is confined to the space of eigenstates of H, *i.e.,* for any B that is representable in the form $B = \sum_{k,l}\beta_{kl}|k >< l|$. Because the operators φ and χ introduce aperiodic functions of φ ,they are not representable in this form. A correct evaluation of $< m|[\chi, H]|n >$ yields the result that corresponds to the canonical commutation relation (2). This is subject of Sec.2.

2 Representation of the commutation relation

We now proceed to evaluate the matrix elements of $< m|[\chi, H]|n >$ directly in terms of the phase variable representation. In terms of φ, the commutation relation (2) is

$$[\varphi, H] = -i\hbar\omega \tag{8}$$

and the phase variable representation of H implied by (8) is

$$H = i\hbar\omega\frac{d}{d\varphi} + f(\varphi) \tag{9}$$

where $f(\varphi)$ is some function of φ. The normalized periodic eigenfunctions of H are

$$< \varphi|n >= \frac{1}{\sqrt{2\pi}}e^{-in\varphi} \tag{10}$$

whose eigenvalues are $n\hbar\omega + f(\varphi)$. In order that the eigenvalues of H be correctly given as $(n + \frac{1}{2}\hbar\omega)$, we must take $f(\varphi) = \hbar\omega/2$. Thus H is given

$$H = i\hbar\omega\frac{d}{d\varphi} + \frac{\hbar\omega}{2} \tag{11}$$

We now express the matrix elements of the commutator as

$$\begin{array}{c} < m|[\chi, H]|n >= -\frac{1}{\omega} < m|[\phi, H]|n > \\ = -(n+\frac{1}{2})\hbar < m|\phi|n > +\frac{1}{\omega} < m|H\phi|n > \end{array} \tag{12}$$

and evaluate the last term as follows

$$\begin{array}{l} \frac{1}{\omega} < m|H\phi|n >= \int_0^{2\pi} < \varphi|m >^* H\varphi < \varphi|n > d\varphi \\ = \frac{\hbar}{2} < m|\phi|n > +i\hbar \int_0^{2\pi} < \varphi|m >^* \frac{d}{d\varphi} < \varphi|n > d\varphi \\ = \frac{\hbar}{2} < m|\phi|n > +i\hbar \int_0^{2\pi} \frac{d}{d\varphi} < \varphi|m >^* \varphi < \varphi|n > d\varphi \\ -i\hbar \int_0^{2\pi} \varphi < \varphi|n > \frac{d}{d\varphi} < \varphi|m >^* d\varphi = i\hbar < \varphi|m >^* \varphi < \varphi|n >|_0^{2\pi} \\ +(m+\frac{1}{2})\hbar < m|\phi|n >= i\hbar + (m+\frac{1}{2})\hbar < m|\phi|n > \end{array} \tag{13}$$

Using

$$\begin{array}{rcl} < m|\phi|n >= \int_0^{2\pi} < \varphi|m >^* \varphi < \varphi|n > d\varphi & = & -\frac{1}{m-n}, \quad if\ m \neq n \\ & = & 0, \qquad if\ m = n \end{array} \tag{14}$$

we finally obtain for the matrix elements of the commutator

$$< m|[\chi, H]|n >= i\hbar + (m-n)\hbar < m|\phi|n >= i\hbar\delta_{mn} \tag{15}$$

Thus, the commutation relation (2) has a *correct* representation in the space of eigenstates of H, despite of the fact that χ *cannot* be represented in that space.

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